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ANISOTROPIC TURBULENCE IN A FLOW OF INCOMPRESSIBLE FLUID BETWEEN ROTATING COAXIAL CYLINDERS*

V.A. BABKIN

The problem of developed turbulent flow of a viscous incompressible fluid between circular coaxial cylinders (Couette flow) is solved. The inner cylinder rotates with constant angular velocity, and the outer cylinder is fixed. The region of flow is divided into two boundary layers adjacent to the cylinders, and the kernel of the flow.

Experimental and theoretical studies have shown /1-4/ that the turbulent flow of a Newtonian fluid near a rigid wall has a structure. The structure consists of an ordered system of boundary vortices which determine, at every point of the stream, a characteristic direction, and at the rigid wall itself the vortices are directed along the streamline. Therefore, the turbulent fluid in the boundary region must be treated as anisotropic /5, 6/ and in /6/ it is assumed that the viscous anisotropy of a turbulent fluid is analogous to the anisotropy of liquid crystals. It was shown in /7/ that the turbulent flow of a Newtonian fluid near a plane wall can, in fact, be described within the framework of the Ericksen-Leslie model /8, 9/ of an oriented fluid, provided that certain additional conditions can be imposed on the defining constants of the model.

In the present paper the turbulent fluid in the boundary regions is regarded as an oriented fluid /7/ and as a viscous fluid with a turbulent viscosity that is constant over the transverse cross-section at the kernel. Unlike existing solutions** (**Dorfman L.A. Hydrodynamic Resistance and Heat Output in Rotating Bodies. Moscow, Fizmatgiz, 1960; Novozhilov V.V. On solving a developed turbulent flow between two coaxial rotating cylinders. Preprint 178, Inst. Problem Mekhaniki, Akad Nauk SSSR, 1981.) the model remains unchanged when changing from the flow between two parallel walls to the flow between rotating cylinders. The solution obtained shows good agreement with experimental data /10/.

Let the space between two infinite smooth coaxial cylinders be filled with an incompressible Newtonian fluid. In order to make comparisons with experimental results /10/ easier, we shall assume that the inner cylinder (of radius R_1) rotates with constant angular velocity ω_1, and the outer cylinder (of radius R_2) is fixed.

We shall use the cylindrical coordinate system r, φ, x. The x axis will be directed along the common axis of the cylinders, and the angle φ will be measured from the direction of rotation of the inner cylinder.

Let the velocity of this rotation be such that the flow of fluid between the cylinders is developed and turbulent. In accordance with the data of /1-4/ we shall divide the region of flow $R_1 \leqslant r \leqslant R_2$ into three subregions: $1 - (R_1 \leqslant r \leqslant r_1)$, $2 - (r_2 \leqslant r \leqslant R_2)$ are the regions of boundary turbulence and $3 - (r_1 \leqslant r \leqslant r_2)$ is the kernel of the flow. In regions 1 and 2 the turbulent fluid has a structure formed by the boundary vortices, and region 3 is regarded as a zone of free turbulence. Since the concept of laminar sublayer is not used in the present paper, the boundaries of the regions 1 and 2 are given without taking into account the thickness of the laminar sublayer at the rigid walls.

We shall seek the velocity distribution separately in each case. In the boundary regions 1 and 2 we shall consider the turbulent fluid as an oriented fluid whose kinematic state at every point is characterized by an averaged velocity $\mathbf{v}(v_r, v_\varphi, v_x)$ and unit orientation vector $\mathbf{n}(n_r, n_\varphi, n_x)$.

When the forces of gravity are neglected and the symmetry of the flow is taken into account, we have $p = p(r)$ and

$$v_r = v_x = 0, \ v_\varphi = v(r) = r\omega(r) \qquad (1)$$
$$n_r = \sin\theta(r), \ n_\varphi = \cos\theta(r), \ n_x = 0$$

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where p is the averaged pressure, θ is the angle between the orientation vector and the stream line and $\omega(r)$ is an unknown function.

Under the assumptions made, the equations of the Ericksen-Leslie model, when they do not become an identity, have the form /9/

$$\frac{d\sigma_{rr}}{dr}+\frac{1}{r}(\sigma_{rr}-\sigma_{\varphi\varphi})-\frac{dp}{dr}+\rho r\omega^2=0 \qquad (2)$$
$$\frac{d\sigma_{\varphi r}}{dr}+\frac{1}{r}(\sigma_{\varphi r}+\sigma_{r\varphi})=0$$

$$\frac{d\mu_{rr}}{dr}+\frac{1}{r}(\mu_{rr}-\mu_{\varphi\varphi})+g_r+\rho_1\omega^2\sin\theta=0 \qquad (3)$$
$$\frac{d\mu_{\varphi r}}{dr}+\frac{1}{r}(\mu_{\varphi r}+\mu_{r\varphi})+g_\varphi+\rho_1\omega^2\cos\theta=0$$

Here ρ is the density, σ_{kl} are the stresses, ρ_1 is a constant characterizing the inertia associated with the rotation of the oriented vector, and μ_{kl}, $\mathbf{g}$ (g_r, g_φ, g_x) are the stresses and the internal volume force responsible for the change in the direction of the orientation vector.

Taking into account relations (1) and the conditions imposed on the Ericksen-Leslie model in /7/, we can write the defining relations for the quantities appearing in Eqs.(2) and (3) in the form

$$\sigma_{rr}=(\mu_1\sin^2\theta+\mu_5)r\omega'\sin\theta\cos\theta \qquad (4)$$
$$\sigma_{\varphi\varphi}=(\mu_1\cos^2\theta+\mu_5)r\omega'\sin\theta\cos\theta$$
$$\sigma_{r\varphi}=\sigma_{\varphi r}=(\mu_1\sin^2\theta\cos^2\theta+\mu_0)r\omega'$$
$$\mu_{rr}=-\mu_{\varphi\varphi}=\Phi\sin\theta\cos\theta \qquad (5)$$
$$\mu_{r\varphi}=-\Phi\sin^2\theta,\ \mu_{\varphi r}=\Phi\cos^2\theta$$
$$\mu_0=\frac{\mu_4+\mu_5}{2},\quad \Phi=k_{22}\left(\frac{\cos\theta}{r}-\sin\theta\frac{d\theta}{dr}\right)$$
$$g_r=g_\varphi=0$$

where μ_1, μ_4, μ_5, k_{22} are the constants of the Ericksen-Leslie model and ω' is the derivative with respect to r.

If we substitute formulas (5) into (3), the latter are transformed into a single equation for the angle θ

$$\sin\theta\cos\theta\frac{d^2\theta}{dr^2}+(\cos^2\theta-2\sin^2\theta)\left(\frac{d\theta}{dr}\right)^2+ \qquad (6)$$
$$\frac{5\sin\theta\cos\theta}{r}\frac{d\theta}{dr}-\frac{\cos^2\theta}{r^2}-\frac{\rho_1}{k_{22}}\omega^2=0$$

Eq.(6) itself is fairly complicated, and is also connected with (2). In order to simplify the problem, we shall make the following assumptions: the inertial effects connected with the rotation of the orientation vector are small and therefore $\rho_1\omega^2=0$; and the angle θ is small, so that $\sin\theta=\theta$, $\cos\theta=1$.

Then Eq.(6) becomes

$$\theta\frac{d^2\theta}{dr^2}+\left(\frac{d\theta}{dr}\right)^2+\frac{5\theta}{r}\frac{d\theta}{dr}-\frac{1}{r^2}=0 \qquad (7)$$

The boundary conditions are formulated separately for regions 1 and 2:

$$\theta(R_i)=0,\ \theta(r_i)=\theta_i,\ i=1,2 \qquad (8)$$

where θ_1 and θ_2 are the values of the angle θ at the boundaries of the flow kernel.

The respective solutions of Eq.(7) with boundary conditions (8) are:

$$2\theta^2=(-1)^{i+1}A_i\left(1-\frac{R_i^4}{r^4}\right)+\ln\frac{r}{R_i} \qquad (9)$$
$$A_i=(-1)^{i+1}\frac{r_i^4[2\theta_i^2-\ln(r_i/R_i)]}{r_i^4-R_i^4}$$

The solutions become simple near the rigid walls, when $|1-r/R_i|\ll 1$,

$$\theta^2=(-1)^{i+1}a_i\left(\frac{r}{R_i}-1\right),\quad a_i=2A_i+\frac{(-1)^{i+1}}{2},\quad i=1,2 \qquad (10)$$

The constants a_1 and a_2 must obviously be positive. The first equation of (2) is used to find $p = p(r)$. Integrating the second equation of (2) under the condition that $\sigma_{\varphi r} = \sigma_{r\varphi}$ (see Eqs.(4)), we obtain the distribution of tangential stresses in regions 1 and 2

$$\sigma_{\varphi r} = -\tau_{wi} R_i^2/r^2, \quad i = 1, 2 \tag{11}$$

where τ_{wi} is the modulus of tangential stress at the wall.

The third equation of (4), within the approximation used, has the form

$$\sigma_{\varphi r} = (\mu_1\theta^2 + \mu_0) r\omega' \tag{12}$$

Equating the right-hand sides of Eqs.(11) and (12) we obtain equations for determining $\omega(r)$ in the boundary regions

$$(\mu_1\theta^2 + \mu_0) r \frac{d\omega}{dr} = -\frac{\tau_{wi} R_i}{r^2}, \quad i = 1, 2 \tag{13}$$

Integrating Eqs.(13) in which θ^2 is given by (10), we obtain the velocity profiles $v = \omega r$ in regions 1 and 2

$$v = B_1 r - K_1 \left[\frac{r}{(\alpha_1 - 1)^3} \ln \frac{r}{r + (\alpha_1 - 1) R_1} + \frac{R_1}{(\alpha_1 - 1)^2} - \frac{R_1^2}{2(\alpha_1 - 1) r}\right] \tag{14}$$

$$v = B_2 r + K_2 \left[\frac{r}{(\alpha_2 + 1)^3} \ln \frac{(\alpha_2 + 1) R_2 - r}{r} + \frac{R_2}{(\alpha_2 + 1)^2} + \frac{R_2^2}{2(\alpha_2 + 1) r}\right]$$

$$K_i = \frac{\tau_{wi}}{\mu_1 a_i}, \quad \alpha_i = \frac{\mu_0}{\mu_1 a_i}$$

Here B_1 and B_2 are the constants of integration of (13) determined, respectively, from the boundary conditions $\omega(R_1) = \omega_1, \omega(R_2) = 0$:

$$B_1 = \omega_1 - K_1 \left[\frac{\ln \alpha_1}{(\alpha_1 - 1)^3} - \frac{1}{(\alpha_1 - 1)^2} + \frac{1}{2(\alpha_1 - 1)}\right] \tag{15}$$

$$B_2 = -K_2 \left[\frac{\ln \alpha_2}{(\alpha_2 + 1)^3} + \frac{1}{(\alpha_2 + 1)^2} + \frac{2}{2(\alpha_2 + 1)}\right]$$

The product $\mu_1\theta^2$ in (12) characterizes the viscous properties of the turbulent fluid governed by its vortex structure, therefore the quantity $\mu_1\theta^2$ can be regarded as the turbulent viscosity. Then we shall have to regard the constant μ_0, irrespective of the inclination of the boundary vortices, as the molecular viscosity. If we assume, as was done in the theory of boundary turbulence /11/, that $\mu_0 = 0$, then for $\alpha_i = 0$ Eqs.(14) will yield the following velocity profiles in regions 1 and 2:

$$v = B_i r + (-1)^i K_i \left[r \ln\left|1 - \frac{R_i}{r}\right| + R_i\left(1 + \frac{R_i}{2r}\right)\right], \quad i = 1, 2 \tag{16}$$

When $\alpha_i = 0$, formulas (15) become meaningless and we must either formulate different boundary conditions for determining the constants B_1 and B_2 or regard the constants B_1 and B_2 are empirical.

If we regard the kernel of the flow $r_1 \leqslant r \leqslant r_2$ as a region of free turbulence with constant turbulent viscosity, then in this case the velocity profile will have the form /11/

$$v = Mr + N/r \tag{17}$$

where M and N are constants independent of the turbulent viscosity and determined from the condition that the velocity is continuous at $r = r_1$ and $r = r_2$.

Below we compare the values at velocity v given by formulas (16) and (17) with the experimental values of velocity given in /10/, in the form of a table (the fluid in question is air, and the radii of the cylinders are $R_1 = 106$ mm and $R_2 = 155$ mm:

$r - R_1$, mm	1	3	5	10	25	40	45	47
$\omega_1 = 178$ sec^{-1}								
v, m/sec. calc.	10.15	9.08	8.69	8.29	7.47	6.83	6.66	6.45
experimental	10.13	9.10	8.70	8.24	7.56	6.90	6.67	6.50
$\omega_1 = 283$ sec^{-1}								
v, m/sec. calc.	16.10	14.61	14.08	13.44	11.96	10.80	10.51	10.12
experimental	16.05	14.60	14.05	13.27	11.95	10.87	10.50	10.10

Calculations using formulas (16) and (17) were carried out for the following values of the parameters: $B_1 = 59.6$ sec, $B_2 = 49.7$ sec, $K_1 = 11.0$ sec, $K_2 = 2.71$ sec, $M = 4.58$ sec, and

$N = 0.90$ m^2/sec for $\omega_1 = 178$ sec, and $B_1 = 100$ sec, $B_2 = 79.3$ sec, $K_1 = 15.7$ sec, $K_2 = 4.71$ sec, $M = 2.17$ sec and $N = 1.53\ m^3$/sec for $\omega_1 = 283$ sec. In both cases we adopted the value $r_1 - R_1 = R_2 - r_2 = 7$mm. The computed and experimental data show good agreement with each other.

It was also found in /10/ that the values of the dimensionless velocity $\bar{v} = v/\omega_1 R_1$ can be found, for any ω_1, on a single graph $\bar{v} = \bar{v}(\xi)$, where $\xi = (r - R_1)/(R_2 - R_1)$. We shall show that the results of this paper also agree with this conclusion.

Let us write B_i and K_i in the form

$$B_i = \beta_i \omega_1,\ K_i = \varkappa_i \omega_1 \tag{18}$$

where β_i and $\varkappa_i$ $(i = 1, 2)$ are dimensionless quantities.

Using the dimensionless quantities $\bar{v}$ and ξ, we rewrite Eqs.(16) as follows:

$$\bar{v} = \beta_1 (1 + \xi h) - \varkappa_1 \left[(1 + \xi h) \ln \frac{\xi h}{1 + \xi h} + \frac{1}{2(1 + \xi h)} + 1\right] \tag{19}$$

$$\bar{v} = \beta_2 (1 + \xi h) + \varkappa_2 \left[(1 + \xi h) \ln \frac{(1 - \xi) h}{1 + \xi h} + (1 + h)\left(1 + \frac{(1 - \xi) h}{2(1 + \xi h)}\right)\right]$$

$$h = (R_2 - R_1)/R_1$$

The values of β_i and $\varkappa_i$, obtained using the values of B_i and K_i, given above, are given in the table. The quantities β_i and $\varkappa_i$, can obviously be regarded as constants, and in this case Eqs.(19) will represent the functions $\bar{v} = \bar{v}(\xi)$ independent of ω_1

ω_1, sec^{-1}	β_1	$\varkappa_1$	β_2	$\varkappa_2$
178	0.33	0.062	0.28	0.015
283	0.35	0.055	0.28	0.017

Since the constants M, N in Eq.(17) can be expressed in terms of B_i and K_i in a linear manner, it follows that formula (17) written in terms of $\bar{v}$ and ξ, also represents the function $\bar{v} = \bar{v}(\xi)$ independent of ω_1. But in this case $\bar{v}$ is independent of ω_1 over the whole of the segment $R_1 < r < R_2$.

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